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Relevant and irrelevant nonlinear Schrödinger equations*

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Abstract. First, we summarize the argument against deterministic nonlinear Schrödinger equations. We recall that any such equation activates quantum non-locality in the sense that that information could be signalled in a finite time over arbitrarily large distances. Next we introduce a deterministic nonlinear Schrödinger equation. We justify it by showing that it is closest, in a precise sense, to the master equations for mixed states used to describe the evolution of open quantum systems. We also illustrate some interesting properties of this equation. Finally, we show that this equation can avoid the signalling problem if one adds noise to it in a precise way. Cases of both discrete and continuous noise are introduced explicitly and related to the density operator evolution. The relevance for the classical limit of the obtained stochastic equations is illustrated on a classically chaotic kicked anharmonic oscillator.

1. Introduction

Quantum mechanics is a linear theory. Physical quantities are represented by linear operators, and the evolution of quantum systems is represented by a linear dynamical equation for the one-dimensional projector P_ψ that represent the state of the quantum system. It is good physics to try to embed quantum mechanics into a wider nonlinear theory in such a way that an upper bound on the nonlinearities can be determined experimentally (see for instance [1]). However, the other fundamental theory of physics, relativity, cannot be ignored in such a program. Indeed, although compatible, quantum mechanics and relativity rely on such different concepts that their relation is strained. In particular, locality and determinism are basic for relativity, whereas quantum mechanics is incompatible with local deterministic hidden variables [2]. This led Shimony to introduce the terminology of 'peaceful coexistence' to describe the peculiar relation between relativity and quantum mechanics [3].

In the next section we recall that any deterministic nonlinear generalization of quantum mechanics breaks this peaceful coexistence: any deterministic nonlinear Schrödinger equation allows one to send signals in a finite time over arbitrarily large distances. Hence, even without experiment, one has good reasons to put an absolute bound of 'zero' on the amount of possible nonlinearities in quantum mechanics! This fact led Weinberg to consider quantum mechanics as a part of the 'final theory' [4]. But, remembering that physics is an experimental science, we shall not follow Weinberg's conclusion. It is perfectly possible that in the more or less near future quantum mechanics and relativity will both be embedded in a wider theory. Let us first stress that in addition to locality, another crucial concept in this reasoning is determinism.

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Despite the negative conclusion of the first section, the second section of this paper presents a nonlinear Schrödinger equation. It is motivated by the fact that it is the nonlinear Schrödinger equation closest to the master equations for density operators that describe the evolution of open quantum systems.

In section 4 we show how the nonlinear Schrödinger equation of section 3 can survive the arguments of section 2. The main idea is to drop determinism. This seems unavoidable to us. Moreover, it seems quite natural to base a theory well known for its stochastic character, like quantum mechanics, on a stochastic evolution equation [5]. The stochastic equations thus obtained are illustrated in several examples. In particular, in section 5 a classically chaotic system is considered and the relevance of the stochastic equation for the classical limit emphasized.

2. All deterministic nonlinear Schrödinger equations are irrelevant

Let us consider a quantum system consisting of two spatially separated subsystems, labelled 'left' and 'right'. Let A and B be two non-commuting linear self-adjoint operators that represent two incompatible physical quantities of subsystem 'left' (i.e. A and B are two observables). Assume that the state of the system is described by a one-dimensional projector (pure state): $P_\psi = |\psi\rangle\langle\psi|$. Assume that a physicist near subsystem 'left' decides to measure the physical quantity A, to record the result α, then to destroy subsystem 'left' (if it has not already been destroyed by the measurement). Subsystem 'right' clearly survives this operation and ends in a certain state. This state $P_\psi(\alpha)$ depends on the results α. Let x_α denote the probability of the result α. For a physicist near subsystem B who ignores the result α, but who knows the procedure carried out by his 'left' colleague, the state of subsystem 'right' is a classical mixture of state $P_\psi(\alpha)$ with probabilities x_α: $\rho_{right} = \sum_\alpha x_\alpha P_\psi(\alpha)$. By classical mixture we mean that the probabilities x_α refer merely to ignorance, like statistical mixtures in classical statistical mechanics. Now, after some finite time t, the actual state of subsystem 'right' will evolve to $P_\psi(\alpha, t)$. Consequently the mixture evolves to: $\rho_{right}(t) = \sum_\alpha x_\alpha P_\psi(\alpha, t)$.

Clearly, the 'left' physicist could also decide to measure the physical quantity B, instead of A, with possible results β that happen with probability y_β. Now, if the corresponding two classical mixtures were distinguishable, then the 'right' physicist could determine which quantity, A or B, the 'left' physicist has measured. Hence he could read a message sent by his colleague. Consequently, in order to avoid arbitrary fast signalling, one concludes that the two classical mixtures at the disposal of the 'right' physicist are indistinguishable:

$$\sum_\alpha x_\alpha P_\psi(\alpha) = \sum_\beta y_\beta P_\psi(\beta)\,. \tag{1}$$

This is called parameter independence [6]. So far we have only used linear quantum mechanics. Next, notice that the 'right' physicist could let his subsystem evolve a finite time. If the evolution $P_\psi(\alpha) \to P_\psi(\alpha, t)$ is linear, the two possible mixtures remain indistinguishable. But in nonlinear quantum mechanics the two mixtures may become distinguishable, $\sum_\alpha x_\alpha P_\psi(\alpha, t) \neq \sum_\beta y_\beta P_\psi(\beta, t)$, and the 'right' physicist could read a message sent by his colleague. Consequently, the equality (1) has to hold for all times t. This implies that the evolution $P_\psi \to P_\psi(t)$ is linear [7].

Finally, in [8] we have proven that any mixture that satisfies (1) can be prepared at a distance in the way described above. From this reasoning we conclude that any deterministic nonlinear Schrödinger equation has predictions in contradiction with relativity and is thus physically irrelevant (as long as a much wider revision of physics is not considered). The

above reasoning has been first sketched in [10, 8], and applied to Weinberg's nonlinear theory in [9].

What did we really use as assumptions? Is the conclusion really inescapable? Several other authors have considered these questions and came to various conclusions [1, 11–15]. Clearly, one could try to modify relativity, but this is outside our scope. It seems that there are three possible paths to follow:

(i) Restrict the set of observables. In this way the density matrix would no longer represent the state of mixtures. Equality (1) would have to be replaced by $\sum_\alpha x_\alpha E(\theta|P_\psi(\alpha)) = \sum_\beta y_\beta E(\theta|P_\psi(\beta))$ for all observables θ, where E stands for the expectation value. Then a similar argument would lead to the conclusion that the map $E(\theta|P_\psi) \to E(\theta|P_\psi(t))$ has to be linear.

(ii) One questionable point is the assumption that once a result α is secured on the 'left', the subsystem 'right' is in the corresponding state $P_\psi(\alpha)$†. But, if not, then there is the possibility to do a measurement on subsystem 'right' before its state is affected by the 'left' measurement result. And, in such a case, how could one explain that any measurement on subsystem 'right' is correctly correlated to the measurement on subsystem 'left'? Note that in a many-world view, the arbitrary fast signalling problem of deterministic nonlinear Schrödinger equations is replaced by signalling among the many worlds [11].

(iii) Determinism is a hidden assumption in all derivations of the Schrödinger equation [16, 5]. But if one takes seriously the idea that quantum physics is not deterministic, then the use of stochastic evolution equations is quite natural. The noise term in such an equation would represent the intrinsic non-determinism of Nature. This is the path we follow, and illustrate it in section 4.

To conclude this section let us emphasize that it would not be enough to note that the nonlinear Schrödinger equations are not relativistic. Indeed, if such an equation is relevant for physics, a relativistic generalization must exist. The predictions of this generalization have to agree with the nonlinear Schrödinger equation whenever all velocities are small. This is precisely the case in the situation discussed above. Hence the generalization would suffer from the same contradiction.

3. An interesting nonlinear Schrödinger equation

Let $\mathcal{L}$ be the generator of a master equation, i.e. of an evolution equation for density operators ρ_t:

$$\dot{\rho}_t = \mathcal{L}\rho_t \,. \tag{2}$$

For instance $\mathcal{L}$ could assume the Lindblad form:

$$\mathcal{L}\rho = -\mathrm{i}[H, \rho] + \sum_{j=1,\ldots,N} \left(L_j \rho L_j^\dagger - \tfrac{1}{2}\{L_j^\dagger L_j, \rho\}\right) \tag{3}$$

where H is the Hamiltonian of the system and the L_j are linear operators that take into account the effect of the environment and $\{\ldots,\ldots\}$ denotes the anticommutator. The Bloch-type master equation (3) assumes a Markov approximation, which is widely used in many areas of physics. Note that this hypothesis imposes some restrictions which are discussed, for instance, in [17]. But this form is not necessary for the presentation of the nonlinear Schrödinger equation in this section.

† Note that it may take a finite time to secure the result α. The crucial point, however, is that once secured, the 'right' subsystem's state has been affected, even if it is far away in space.

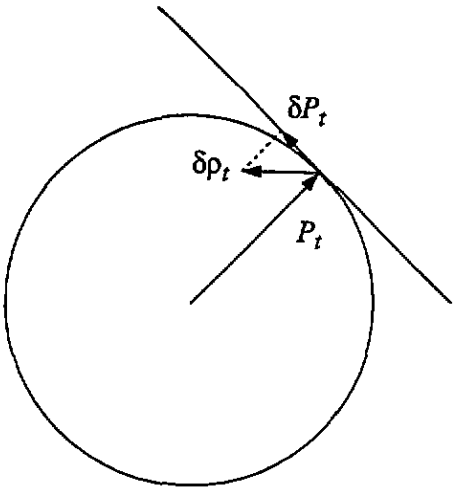

Figure 1. Illustration of the relation between the density matrix evolution equation (2) and the nonlinear Schrödinger equation (4) on the Poincaré sphere for spin-$\frac{1}{2}$ systems. The points on the sphere P_t represent pure states, whereas the inner points ρ_t label mixed states. The flow $\delta\rho_t$ represents the evolution of density matrices. The corresponding nonlinear Schrödinger equation is obtained by projecting the flow $\delta\rho_t$ onto the plane tangent to the sphere: $\delta P_t = (P_t \wedge \delta\rho_t) \wedge P_t$. For operators, the vector product $\wedge$ translates into i times the commutator, see (A4). For simplicity only a section of the sphere is shown.

In this section we consider the following nonlinear Schrödinger equation which is naturally associated to the master equation (2):

$$\dot{\psi}_t = (\mathcal{L}P_{\psi_t} - \langle \mathcal{L}P_{\psi_t}\rangle_t)\psi_t \tag{4}$$

where $\langle \mathcal{L}P_{\psi_t}\rangle_t = \langle\psi_t|\mathcal{L}P_{\psi_t}|\psi_t\rangle$, and $P_{\psi_t} = |\psi_t\rangle\langle\psi_t|$ denotes the one-dimensional projector associated with the normalized vector ψ_t. In terms of this projector, the above equation reads

$$\dot{P}_{\psi_t} = \{\mathcal{L}P_{\psi_t}, P_{\psi_t}\} - 2P_{\psi_t}\,\mathcal{L}P_{\psi_t}\,P_{\psi_t}\,. \tag{5}$$

Equations (4) and (5) are equivalent. It seems that their first presentation is due to Diosi [18]. Figure 1 illustrates in which sense this nonlinear Schrödinger equation is closest to the master equation (2): one projects the flow generated by (2) onto the manifold of pure states. In the appendix we make this more precise and derive the nonlinear Schrödinger equation (5).

Let us have a closer look at the case where the sum in (3) reduces to one term with a self-adjoint environment operator, $L_1 = A = A^\dagger$ and where the Hamiltonian is neglected:

$$\dot{\psi}_t = a_t(A - a_t)\psi_t - \tfrac{1}{2}(A^2 - \langle A^2\rangle_t)\psi_t \tag{6}$$

where $a_t = \langle A\rangle_t$.

Several comments are in order:

(i) Equation (6) is truly nonlinear. By truly nonlinear we mean that the nonlinearity is not merely a normalization factor.

(ii) Equation (6) is invariant under the shift $A \to A + \lambda$, where λ is any real number. (If λ is imaginary, then this 'shift' produces a Hamiltonian term).

(iii) The recalling $A \to \lambda A$ in equation (6) corresponds merely to a recalling of time.

By reordering the terms in (6) one obtains

$$\dot{\psi}_t = -\tfrac{1}{2}((A - a_t)^2 - \langle (A - a_t)^2\rangle_t)\psi_t\,. \tag{7}$$

This form of the equation shows that the solutions tend to minimize $(A - a_t)^2$. And, indeed, all the eigenstates of A are *stable* stationary solutions of (7). Note that there are also some other stationary solutions, but only the eigenstates are stable.

The study of (4) is also interesting in the case of non-self-adjoint environment operators L_j. Consider, for instance, the case of a harmonic oscillator in a thermal bath: $L_1 = \gamma_1 a$

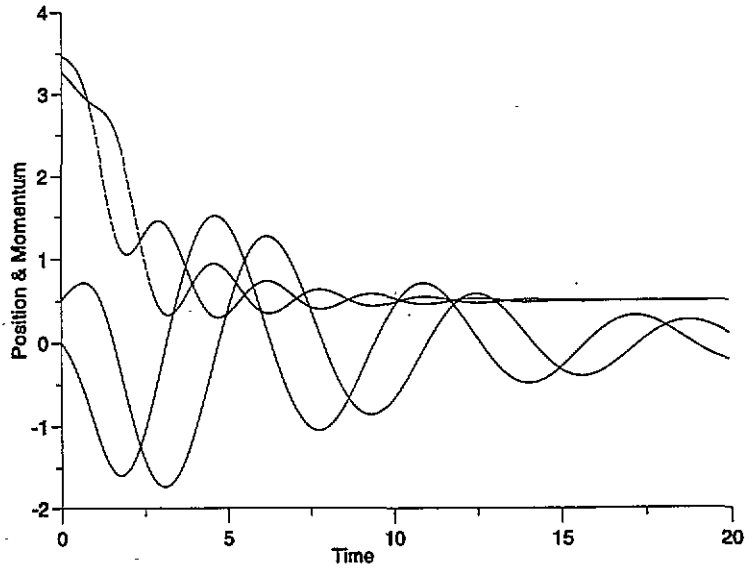

Figure 2. Illustration of the solution of the nonlinear Schrödinger equation (4) in the case of a harmonic oscillator in a heat bath: $H = a^\dagger a$, $L_1 = \sqrt{2}\frac{3}{8}a$, $L_2 = \sqrt{2}\frac{1}{8}a^\dagger$ and initial state $\psi_0 = |3\rangle + \frac{1}{10}(|2\rangle + |4\rangle)$. The full curves represent the expectation values of position and momentum, while the broken curves represent the square of the corresponding standard deviations. Clearly the latter tend to their minimum value $\frac{1}{2}$. Accordingly, the state tends to a coherent state.

and $L_2 = \gamma_2 a^\dagger$, where a and $a^\dagger$ are the usual annihilation and creation operators. In this case any initial state tends to a coherent state labelled by a complex number $\alpha_t = \langle a \rangle_t$. This complex number follows the equation

$$\dot{\alpha}_t = \frac{\gamma_2^2 - \gamma_1^2}{2}\alpha_t \,. \tag{8}$$

Figure 2 illustrates this case with $\gamma_1 = \sqrt{2}\frac{3}{8}$, $\gamma_2 = \sqrt{2}\frac{1}{8}$ and a Hamiltonian $H = a^\dagger a$. The full curves represent the expectation values of position and momentum, while the broken curves represent the squares of the corresponding standard deviations. Clearly, the latter tend to their minimum value $\frac{1}{2}$.

A remarkable property of (4) is the following. For any Hamiltonian quadratic in a and $a^\dagger$ (that is, quadratic in position and momentum), and any environment operators L_j linear in a and $a^\dagger$ (linear in position and momentum) and any initial condition, the solution of (4) corresponding to the Liouvilian (3) tends to a Gaussian state, that is a state of minimum uncertainty $\Delta p \Delta q = \frac{1}{2}$, known as squeezed states.

Figure 3 illustrates the case of a linearly damped harmonic oscillator: $H = \omega a^\dagger a + i\frac{k^2}{2}((a^\dagger)^2 - a^2)$ and $L_1 = \sqrt{2}ka$. According to the master equation (2): $\langle \dot{p} \rangle = -\omega\langle q \rangle - 2k^2\langle p \rangle$ and $\langle \dot{q} \rangle = \omega\langle p \rangle$. According to the nonlinear Schrödinger equation (4) the expectation values follow more complicated trajectories, but figure 3 illustrates that damping is clearly present.

Figure 4 illustrates a pumped and damped nonlinear oscillator. In this case the state does not tend to a Gaussian state.

The most fundamental difference between the master equation (2) and the nonlinear Schrödinger equation (4) is that the second one is deterministic, whereas the first one is

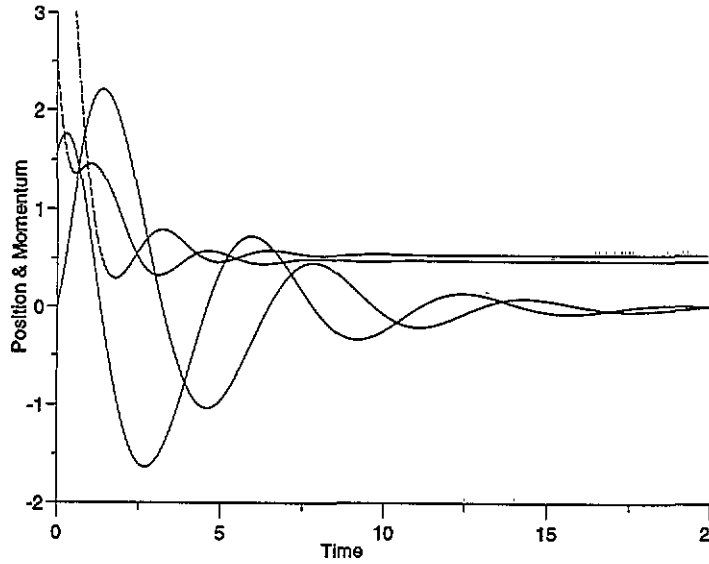

Figure 3. Same as figure 2, but for a linearly damped harmonic oscillator (Caldeira–Leggett–Diòsi master equation [39]): $H = a^\dagger a + \frac{i}{8}((a^\dagger)^2 - a^2)$, $L_1 = \frac{1}{\sqrt{2}}a$ and initial state $\psi_0 = |4\rangle + i|5\rangle$.

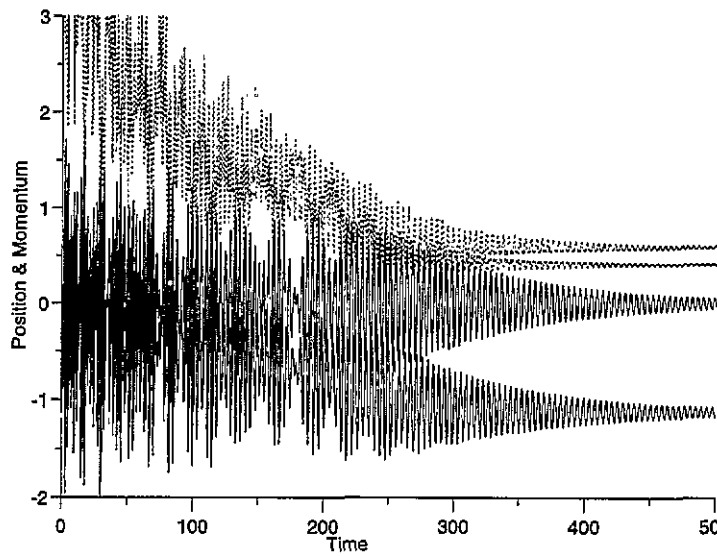

Figure 4. Same as figure 2, but for a pumped and damped nonlinear Hamiltonian: $H = a^\dagger a + \frac{1}{4}(a^\dagger)^2 a^2 + i(a^\dagger - a)$, $L_1 = \frac{\sqrt{2}}{10}a$ and initial state $\psi_0 = |4\rangle$.

stochastic in the sense that it turns pure states into mixed states. In the next section we shall see how to add randomness to the nonlinear Schrödinger equation in such a way as to recover the master equation.

4. How to make the interesting nonlinear Schrödinger equation relevant

In this section we illustrate how discrete or continuous noise added to the nonlinear Schrödinger equation (4) presented in the previous section solves the problem posed in section 2, while keeping the most interesting properties of that equation. Actually we shall limit our discussion to the Markovian case (3). As we shall see, the main consequences are:

(i) The evolved states averaged over the noise reproduce exactly the original master equation (2). Since the noise cannot be controlled, the physicist has access only to the averaged state. Consequently, mixtures that are indistinguishable remain indistinguishable. The signalling problem, summarized in section 2, is thus avoided.

(ii) The noise drives the system towards stable solutions. Depending on the Hamiltonian H and the environment operators L_j, the stable solutions may be, for instance, Gaussian states localized in phase space, or eigenvector, as in measurement situations. Accordingly, in a natural way the noise introduces a reduction of the wavefunction without an *ad hoc* postulate.

(iii) The probabilities of reduction to an eigenstate are precisely equal to the probability of occurrence of the corresponding eigenvalue as predicted by quantum mechanics.

(iv) The individual realizations provide a description of the evolution of individual open quantum systems.

Let us first consider the case of discrete Poissonian noises $\mathrm{d}N_{j,t}$:

$$\mathrm{d}P_{\psi_t} = \dot{P}_{\psi_t}\,\mathrm{d}t + \sum_j (P_{j,t} - P_{\psi_t})\,\mathrm{d}N_{j,t} \tag{9}$$

where $\dot{P}_{\psi_t}$ is defined by (5), for $\mathcal{L}$ given by (3). For each time increment $\mathrm{d}t$, $\mathrm{d}N_{j,t} = 0$ or 1 (0 or 1 jump happen). Hence the Itô table reads: $\mathrm{d}N_{j,t}\,\mathrm{d}N_{k,t} = \delta_{j,k}\,\mathrm{d}N_{j,t}$ and $\mathrm{d}N_{j,t}\,\mathrm{d}t = 0$. Such a process is characterized by its mean value, denoted $M\,\mathrm{d}N_{j,t}$, proportional to $\mathrm{d}t$ which determines the probability density of the jumps:

$$\sum_j M\,\mathrm{d}N_{j,t} = -M\langle \mathcal{L}P_{\psi_t}\rangle\,\mathrm{d}t\,. \tag{10}$$

The intuitive meaning of equation (9) should be clear. Most of the time the $\mathrm{d}N_{j,t}$ vanish and (9) is just the nonlinear Schrödinger equation presented in the previous section. From time to time one $\mathrm{d}N_{j,t}$ takes the value one and dominates (9). This corresponds to a jump $P_{\psi_t} \to P_{j,t}$ (the probability that more than one $\mathrm{d}N_{j,t}$ takes the value one is negligible). Note that states onto which the system can jump, $P_{j,t}$, depend on the actual state P_{ψ_t}, hence on time.

One can now compute the mean of (9):

$$M\,\mathrm{d}P_{\psi_t} = M(\{\mathcal{L}P_{\psi_t}, P_{\psi_t}\} - 2P_{\psi_t}\,\mathcal{L}P_{\psi_t}\,P_{\psi_t})\,\mathrm{d}t + \sum_j M(P_{j,t}\,\mathrm{d}N_{j,t}) - \sum_j M\,\mathrm{d}N_{j,t}\,P_{\psi_t}\,\mathrm{d}t$$

$$= M\mathcal{L}P_{\psi_t}\,\mathrm{d}t + \sum_j M(P_{j,t}\,\mathrm{d}N_{j,t}) - R\,\mathrm{d}t$$

where the operator R is defined as: $R = M(1 - P_{\psi_t})\mathcal{L}P_{\psi_t}(1 - P_{\psi_t})$. Consequently, the original master equation (2) is recovered provided that the last two terms cancel each other: $\sum_j M(P_{j,t}\,\mathrm{d}N_{j,t}) = R\,\mathrm{d}t$. In words, the mixture of states $P_{j,t}$ with weights $M\,\mathrm{d}N_{j,t}/\mathrm{d}t$ corresponds to the non-normalized 'density matrix' R. If $\mathcal{L}$ assumes the Lindblad form (3), the operator R is always positive and the above construction is well defined. In that case one can, following Diósi [18, 19], assume the $P_{j,t}$ orthogonal to each other and thus obtain a unique jump process associated with any Lindblad master equation. Note, however, that

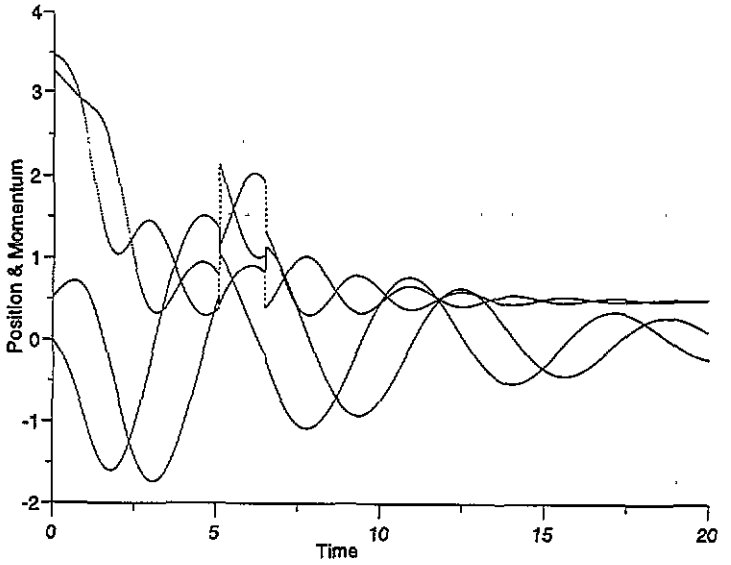

Figure 5. Same as figure 2, but for the stochastic jump (9). Two jumps are clearly displayed.

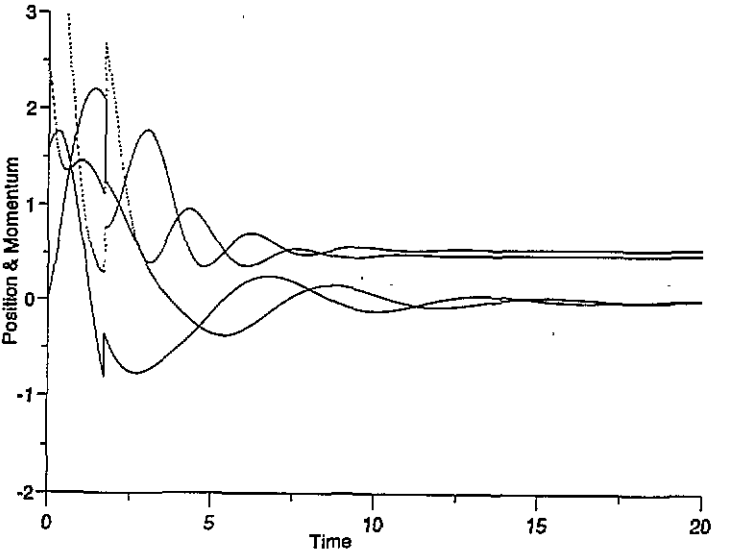

Figure 6. Same as figure 3, but for the stochastic jump (9).

for some peculiar cases, in particular for non-Markovian master equation, the operator R is not necessarily positive and this construction does not work.

Figures 5–7 illustrate solutions of the stochastic jump in equation (9) for the same systems as considered for the nonlinear Schrödinger equation in figures 2–4, respectively. Only a few jumps happen, but they are just enough to supplement the deterministic nonlinear Schrödinger equation (4) with the randomness necessary to be compatible with the density operator evolution. In the case of the nonlinear oscillator this randomness speeds up the localization.

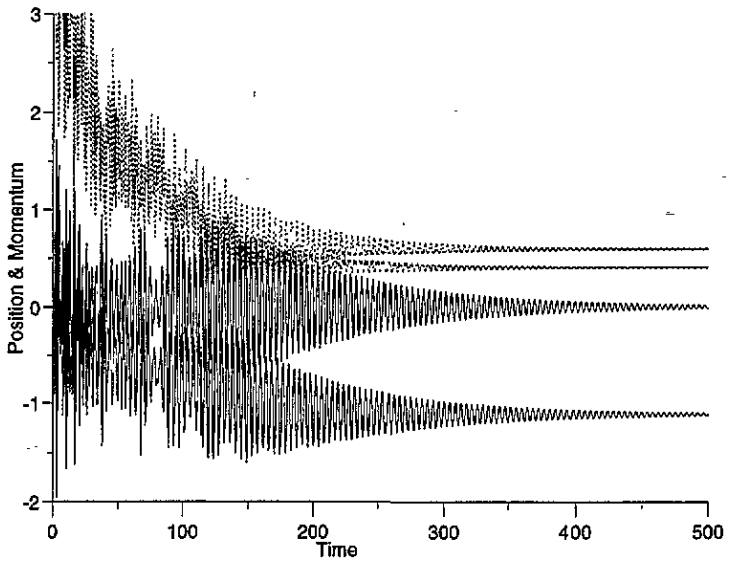

Figure 7. Same as figure 4, but for the stochastic jump (9). Note that localization is faster than in the deterministic case.

This stochastic process, called the 'orthogonal jump process' by Diósi [18, 19], has recently been rediscovered by Milburn and co-workers [20] in the context of optimum quantum measurement (i.e. that requiring the least information to keep track of the system). Note that it is different from the jump process introduced by Carmichael [21] and by Dalibard and co-workers [22] under the name 'Monte Carlo wavefunction'.

In the case of a Lindblad master equation with a single environment operator L, the stochastic equation (9) takes the following form, in terms of the normalized state vector ψ_t:

$$\mathrm{d}\psi_t = \langle L^\dagger\rangle_{\psi_t}(L - \langle L\rangle_{\psi_t})\psi_t\,\mathrm{d}t - \tfrac{1}{2}(L^\dagger L - \langle L^\dagger L\rangle_{\psi_t})\psi_t\,\mathrm{d}t + \left(\frac{L - \langle L\rangle_{\psi_t}}{\sqrt{\langle L^\dagger L\rangle_{\psi_t} - \langle L^\dagger\rangle_{\psi_t}\langle L\rangle_{\psi_t}}} - 1\right)\psi_t\,\mathrm{d}N_t\,.$$

Accordingly, the nonlinear Schrödinger evolution is interrupted from time to time, with a mean frequency $M\,\mathrm{d}N_t = (\langle L^\dagger L\rangle_{\psi_t} - \langle L^\dagger\rangle_{\psi_t}\langle L\rangle_{\psi_t})\,\mathrm{d}t$. During these interruptions the state vector ψ_t jumps to the orthogonal vector $(L - \langle L\rangle_{\psi_t})\psi_t$.

We consider now the case of continuous Gaussian noises $\xi_{j,t}$, for master equations of the Lindblad form (3):

$$\mathrm{d}P_{\psi_t} = \dot{P}_{\psi_t}\,\mathrm{d}t + \sum_j (X_j P_{\psi_t} \circ \mathrm{d}\xi_{j,t} + P_{\psi_t} X_j^\dagger \circ \xi_{j,t}^*) \tag{11}$$

where the X_j are nonlinear operators and the circle in front of the $\mathrm{d}\xi_{j,t}$ indicates that (11) is a Stratonovich stochastic equation [23]. From a practical point of view 'Stratonovich stochastic equation' means that the usual rule of ordinary analysis can be applied. But, in order to compute mean values, the Itô form of stochastic equations is more convenient. Indeed, in the latter form the stochastic increments $\mathrm{d}\xi_{j,t}$ are 'non-anticipating', hence $M(X\,\mathrm{d}\xi_{j,t}) = 0$ for all X. The Stratonovich and Itô equations are related through the simple rule: $X \circ \mathrm{d}Y = X\,\mathrm{d}Y + \frac{1}{2}\,\mathrm{d}X\,\mathrm{d}Y$ (where the lack of circle implies the Itô form). Consequently, since the master equation (2) is quadratic in the environment operators L_j,

the X_j must be linear in the L_j. Moreover, since the trace of P_{ψ_t} is constant, the expectation values of the X_j must vanish. Hence

$$X_j = L_j - \langle L_j \rangle_{\psi_t} . \tag{12}$$

Note that a possible multiplicative constant could be absorbed in the $d\xi_{j,t}$. The corresponding stochastic evolution equation for the normalized state vector ψ_t reads

$$d\psi_t = \dot{\psi}_t \, dt + \sum_j (L_j - \langle L_j \rangle_{\psi_t}) \psi_t \circ d\xi_{j,t} \tag{13}$$

where $\dot{\psi}_t$ is given by (4). A straightforward computation shows that the mean, $M P_{\psi_t}$, of the pure states P_{ψ_t} over the noises $\xi_{j,t}$ follows the master equation (2), provided the $d\xi_{j,t}$ are complex Wiener processes of zero mean and correlations:

$$M(d\xi_{j,t} \, d\xi_{k,t}) = 0 \qquad M(d\xi^*_{j,t} \, d\xi_{k,t}) = \delta_{j,k} \, dt . \tag{14}$$

The equations (13) and (14) describe a continuous evolution similar to Brownian motion, but in Hilbert space. The dissipation, described by the nonlinear Schrödinger equation, and the fluctuation balance each other in such a way that the mean values follow precisely the evolution corresponding to the master equation. It has been first introduced in [24, 25]. Similar equations, but with real Wiener processes, were presented in [8, 18, 26–28]. More general equations, mixing Gaussian and Poissonian noise, can also be constructed.

In [25] we emphasized the use of (13) for practical computations based on a Monte Carlo algorithm. We proved general localization theorems and illustrated the physical picture and insight provided by this state diffusion model. Finally, let us mention some applications and tests of the model:

(i) in [29] QSD is used to describe a quantum jump experiment;

(ii) in [30] QSD is applied to some nonlinear optical processes;

(iii) in [31] QSD and quantum jump simulations are compared for two-photon processes;

(iv) in [32] the 'phase space picture' of QSD and quantum jumps are compared;

(v) in [33] the approach to thermal equilibrium of harmonic oscillators is investigated numerically;

(vi) in [34] QSD is applied to an open angular system, such as a quantum capacitor or rotor;

(vii) in [35, 36] some preliminary results on the relation to quantum chaos are presented;

(viii) in [37] some explicit solutions are presented;

(ix) in [38] the Heisenberg picture is investigated, and algorithms for multi-time expectation values and correlation functions are presented. Note that all the constructions presented in the present paper, i.e. deterministic equation, continuous Gaussian noise, orthogonal jump process, can also be carried out for the transition operators considered in [38].

Figures 8–10 present sample solutions of the stochastic equation (13) for the same cases as illustrated for the deterministic cases in figures 2–4, respectively. The solutions are clearly similar, but the additional noise corrects precisely the defects of the deterministic case. For instance the harmonic oscillator in a heat bath at finite temperature (figures 2 and 8) is damped down to the ground state in the deterministic case, whereas in the stochastic case it keeps oscillating around the ground state, as it should for a physical model of an oscillator at finite temperature. Comparing figures 4 and 10 which present a nonlinear oscillator is also instructive: the noise has two effects. First it localizes the oscillator state in phase space, next it keeps the state away from a non-physical stationary solution.

The relation between the stochastic equation presented in this section and the master equation (3) is unique [41]. Indeed the assumption of continuous Markovian diffusion leads

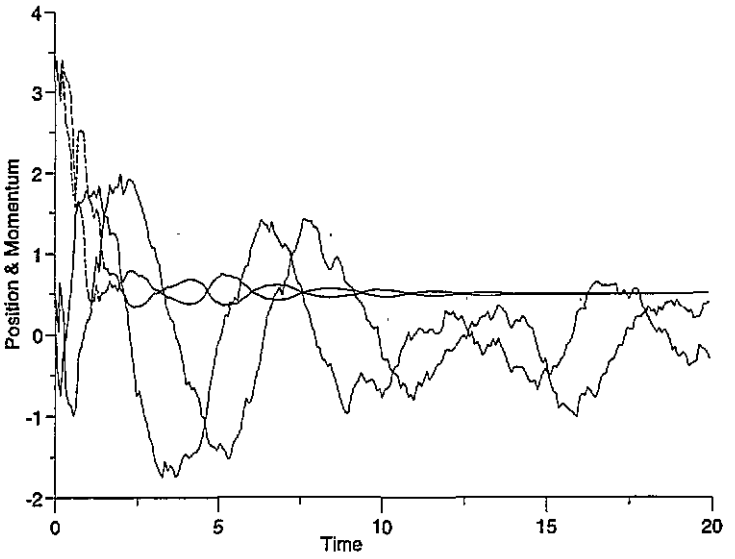

Figure 8. Same as figure 2, but for the continuous stochastic equation (13). Note that the state ψ_t tends also to a coherent state, but that the latter fluctuates forever, as it should since the temperature is finite.

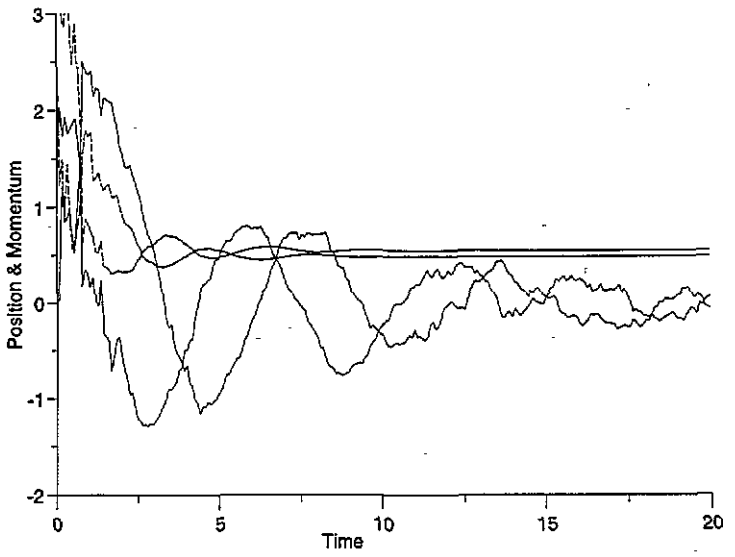

Figure 9. Same as figure 3, but for the continuous stochastic equation (13). Note that the state ψ_t tends also to a Gaussian state, but that the latter fluctuates forever. On average, one has linear damping: $M\langle\dot{p}\rangle = -M\langle q\rangle - \frac{1}{2}M\langle p\rangle$ and $M\langle\dot{q}\rangle = M\langle p\rangle$.

to equation (13), whereas piecewise continuous trajectories lead to equation (9). Hence classical ensembles of pure state trajectories (continuous or not) are uniquely associated to the solution of the master equation (3) (given the initial condition). This is in sharp contrast to the kinematic case, as there are many classical ensembles of pure states corresponding to a given density matrix.

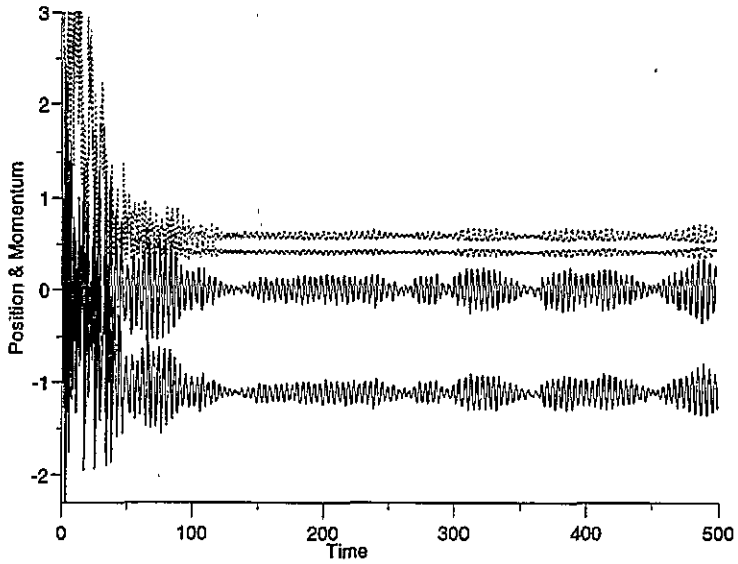

Figure 10. Same as figure 4, but for the continuous stochastic equation (13). Note that localization is much faster than in the deterministic case or the jump case, but that the state ψ_t never reaches a stationary shape (it keeps breathing).

In this section we saw that nonlinear Schrödinger equations can be made much more interesting and relevant to physics by taking into account the fluctuations that are well known to accompany dissipation.

5. Quantum chaos and the classical limit

Most classical dynamical systems have chaotic regions in the phase space, at least for some values of the parameters. In contrast, quantum systems do not exhibit chaos, at least in the usual sense of exponential sensitivity to initial conditions. This is a difficulty for the correspondence principle and for the classical limit [40]. Now, since both stochastic equations presented in the previous section do have localized wavefunctions as solutions, one may expect that the Ehrenfest theorem applies and that, whenever the localization is strong enough, the localized stochastic wavefunctions exhibit chaos. This phenomenon of localization has been investigated in [25, 43]. In [42] Percival argued that the localized quantum states do 'appear to classical eyes like phase space points'. Recently, Spiller nicely illustrated that phenomenon on a simple kicked anharmonic oscillator [35]. The classical system is known to have a strange attractor, whereas the density matrix (not surprisingly) exhibits no chaos [44]. Now, the solutions of the continuous stochastic equation (13) exhibit the strange attractor, see figure 13 (this figure is very similar to the one obtained by Spiller, we include it for completeness and because it has been obtained with independent software on a PC). Clearly, the attractor appears with some cloudiness, but this was to be expected for a quantum system. Also the periodic point around $(-5.88, 5.88)$ appears, with random transitions between it and the attractor. Spiller's example is a beautiful illustration of the insight that stochastic wavefunctions can provide, in particular with the connection to the classical limit.

Figures 11 and 12 present the kicked anharmonic oscillator as described by the nonlinear

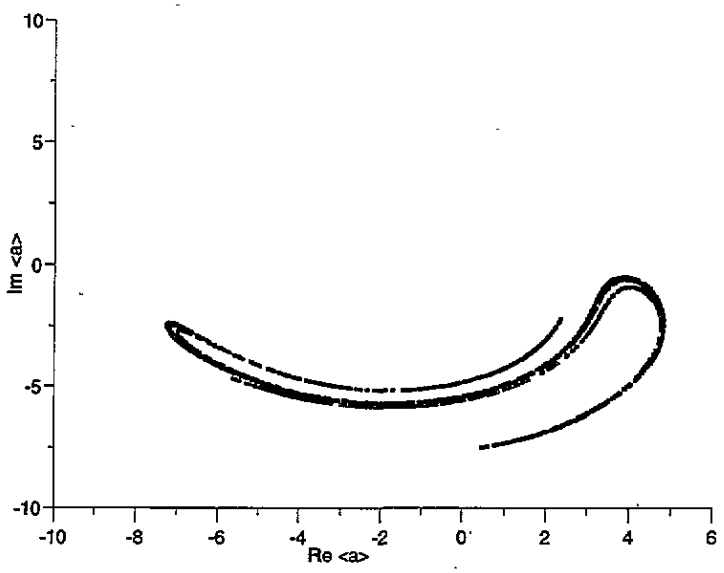

Figure 11. The nonlinear Schrödinger equation (4) applied to the kicked anharmonic oscillator: $H = 0.004(a^\dagger)^2a^2 + F(t)\mathrm{i}(a^\dagger - a)$, $L = \sqrt{0.1}a$, where $F(t)$ is a periodic step function taking value 0 for periods of 5 time units and value 2 for periods of length 4.9. The time increment was chosen as 0.01, the initial state was the ground state of the (unkicked) oscillator. The Hilbert space was truncated to 56 dimensions with a moving basis formed by the eigenstates of the harmonic oscillators $|n_0\rangle$ to $|n_0+55\rangle$, every 20 time steps n_0 was adapted to the actual quantum state. A Poincaré section (of period 9.9) is shown. This figure is indistinguishable from the strange attractor obtained from the corresponding classical dynamical equations [35].

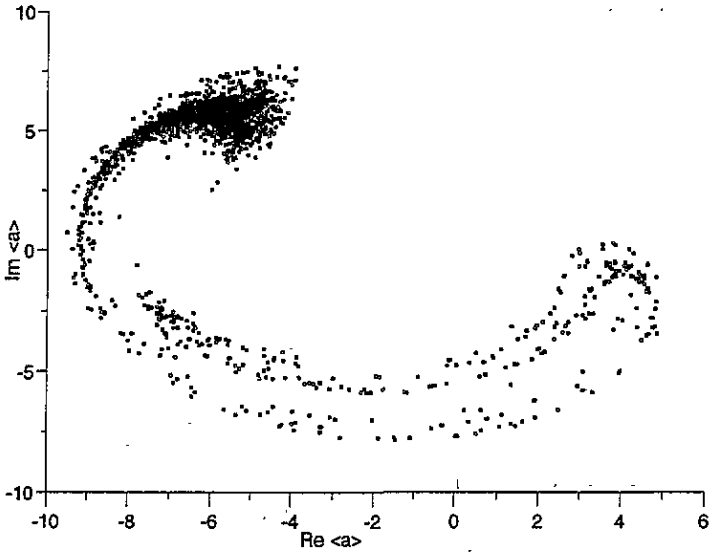

Figure 12. Same as figure 11, but for the stochastic jump (9) applied to the kicked anharmonic oscillator. The strange attractor and the fixed point both appear quite clearly, with random transitions between them. For this example only, the truncation at 56 dimensions was insufficient; we used a dimension of 160.

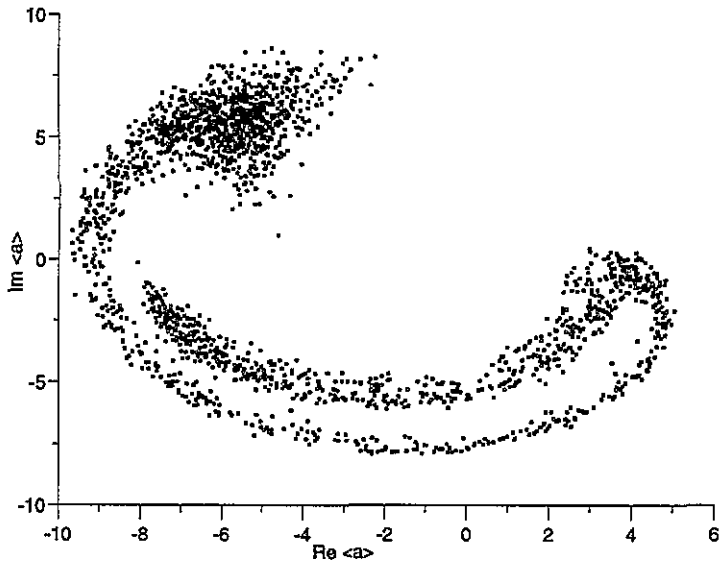

Figure 13. Same as figure 11, but for the continuous stochastic equation (13) applied to the kicked anharmonic oscillator. The strange attractor and the fixed point both appear quite clearly, with random transitions between them.

Schrödinger equation (4) and the stochastic jump equation (9), respectively. In the deterministic case, the attractor appears particularly clearly. The periodic point is there too (though it is not displayed). This should not surprise the careful reader, since a maximally localized state (in phase space) is a coherent state $|\alpha_t\rangle$ for which the noise term in (4) vanishes: $(a-\langle a\rangle_{\alpha_t})|\alpha_t\rangle = 0$. Similarly, for the jump equation (9) the probability of a jump vanishes for coherent states. Note that at finite temperature the noise term never vanishes.

Consequently, the strange attractor of the classical kicked anharmonic oscillator is obtained within the quantum state diffusion model in the classical limit.

6. Conclusion

First, we summarized the argument based on quantum non-locality against nonlinear Schrödinger equations. This argument is based on the concept of preparation at a distance: the possibility to prepare different mixtures of pure states while acting only on a 'sister system' at a distance. The different mixtures correspond to the same density operator, but a nonlinear Schrödinger equation would allow one to distinguish among them. We emphasize that the projection postulate plays little role in this argument: only some form of reduction is assumed. If one assumes that there is no reduction at all, one enters the 'universe' of 'many worlds' in which the argument translates into 'communications between different world components' (whatever that means in a completely deterministic physics).

Next, we considered open quantum systems. With the aim of describing individual open systems we derived and studied the nonlinear Schrödinger equation closest to the usual master equation for density operators commonly used in the quantum theory of open systems.

Finally, we emphasized that noise can be added in natural ways to this nonlinear Schrödinger equation. In this way we recover the master equation as the mean of random

pure states. Hence the predictions of quantum mechanics for open systems are recovered, while keeping a description of individual systems. Moreover, we saw that the correct classical limit obtains for a chaotic kicked anharmonic oscillator.

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Appendix

In this appendix we show that the nonlinear Schrödinger equation presented in section 3 is indeed closest to the master equation (2). Let δP_ψ denote the change of the one-dimensional projector representing the state of the system over a small time increment δt. By 'closest' we mean that

$$\mathrm{Tr}((\mathcal{L}\rho\delta t - \delta P_\psi)^2) \text{ is minimum}. \tag{A1}$$

In addition to minimizing the above trace, δP_ψ has also to satisfy some conditions in order that $P_\psi + \delta P_\psi \delta t$ is still a one-dimensional projector (i.e. $P^\dagger = P$, $P^2 = P$ and $\mathrm{Tr}(P) = 1$). These additional conditions imply that there is a self-adjoint operator X such that $\delta P_\psi = [P_\psi, [P_\psi, X]]$. After some algebra, one obtain for the expression (A1) the convenient form

$$\mathrm{Tr}((\mathcal{L}\rho\delta t)^2) - \mathrm{Tr}((\{2\mathcal{L}\rho\delta t - X, X\})P_\psi) + 2\,\mathrm{Tr}(XP_\psi)\,\mathrm{Tr}((2\mathcal{L}\rho\delta t - X)P_\psi)\,. \tag{A2}$$

Now, define $Y = \mathcal{L}\rho\delta t - X$. The expression (A2) simplifies further:

$$\mathrm{Tr}((\mathcal{L}\rho\delta t)^2) - 2(\mathrm{Tr}((\mathcal{L}\rho\delta t)^2 P_\psi) - \mathrm{Tr}(\mathcal{L}\rho\delta t P_\psi)^2) + 2(\mathrm{Tr}(Y^2 P_\psi) - \mathrm{Tr}(Y P_\psi)^2)\,. \tag{A3}$$

This expression is minimum whenever the operator Y admits ψ as an eigenvector. Consequently

$$\delta P_\psi = [P_\psi, [P_\psi, \mathcal{L}\rho]]\delta t \tag{A4}$$

which indeed corresponds to the nonlinear Schrödinger equation (5).

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